Intersecting Black Branes
in Strong Gravitational Waves

Li-Ming Cao[a], Oleg Evnin[b] and Nobuyoshi Ohta[a]

[a] *Department of Physics, Kinki University*
Higashi-Osaka, Osaka 577-8502, Japan

[b] *Institute of Theoretical Physics, Academia Sinica*
Zhōngguāncūn dōnglù 55, Beijing 100190, China

caolm@phys.kindai.ac.jp, eoe@itp.ac.cn, ohtan@phys.kindai.ac.jp

ABSTRACT

We consider intersecting black branes with strong gravitational waves propagating along their worldvolume in the context of supergravity theories. Both near-horizon and space-filling gravitational wave modes are included in our ansatz. The equations of motion (originally, partial differential equations) are shown to reduce to ordinary differential equations, which include a Toda-like system. For special arrangements of intersecting black branes, the Toda-like system becomes integrable, permitting a more thorough analysis of the gravitational equations of motion.

1 Introduction

In a previous publication [1], we have derived a class of gravitational solutions corresponding to a single black brane with strong gravitational waves propagating along its worldvolume. The algebraic structure responsible for solving the equations of motion in that case appears to be quite general and applies to the case of multiple intersecting branes as well. The reason for pursuing such a generalization, which we shall presently undertake, is twofold. First, intersecting brane solutions with (near-horizon) waves propagating on them appear in the context of string theory studies of black hole evaporation [2]. Second, the relative complexity of the intersecting brane case forces us to re-examine the qualitative structure of the equation of motion and obtain a more explicit and general picture of their properties, including a general counting of independent non-linear gravitational wave amplitudes.

The solutions we construct exactly describe interactions of essentially non-linear structures (black branes and strong gravitational waves). Not many solutions of this type (black objects in time-dependent backgrounds) are known. Some analogues featuring ordinary (rather than light-like) time dependences are black holes in Friedmann-Robertson-Walker [3, 4] and de-Sitter [5, 6] spacetimes, as well as supergravity p-branes embedded in dilaton cosmologies [7]. In our present solutions, the black branes are embedded into strong gravitational waves (rather than time-dependent cosmologies), and a large number of arbitrary functions of the light-cone time describing the non-linear gravitational wave amplitudes are present in the corresponding solutions.

Strong gravitational waves in flat space-time are known to be described by the metric (see, e.g., Appendix A of [8])

$$ds^2 = -2du\, dv + K_{ij}(u)x^i x^j du^2 + (dx^i)^2, \tag{1}$$

where $K_{ij}(u)$ represent the profiles of different polarization components of the wave. In pure gravity, $K_{ij}(u)$ is constrained by $K_{ii} = 0$, giving the same number of polarizations as in linearized theory (a traceless symmetric tensor in $D-2$ dimensions, with D being the number of dimensions of space-time). If a dilaton is present, K_{ii} does not vanish and is related to the dilaton, which gives an additional independent polarization component. The metric (1) is given in the so-called Brinkmann coordinates (which are not prone to coordinate singularities and hence useful for global considerations). By a u-dependent rescaling of x^i, it can be brought to the so-called Rosen form, in which the metric only depends on u, making the planar nature of the wave front manifest.

Gravitational solutions of increasing generality describing black branes embedded in plane waves (1) have been constructed in a series of publications over the last few years. Thus, solutions for supersymmetric p-brane-plane-wave configurations with specific choices of the plane wave profile have been constructed in [9]–[12]. A simple class of such solutions (for which the dependences on the distance from the brane and on the light-cone time factorize in the metric) has also been considered in [13] in conjunction with AdS/CFT approach to light-like 'cosmologies'. Extremal supersymmetric solutions with an arbitrary gravitational wave profile were constructed in [14], and solutions with an arbitrary profile featuring extremal intersecting p-branes were constructed in [15]. Non-extremal solutions for black branes embedded in plane waves were constructed in [1].

These latter solutions feature strong gravitational waves localized around the black brane worldvolume, as well as space-filling waves of the type (1). We shall presently generalize the considerations of [1] to the case of intersecting branes. (For some related literature, see [16]–[24].)

We have already remarked in [1] on the motivation for investigating our present class of solutions. Black-brane-plane-wave configurations appear in the context of string theory investigations of black hole evaporation [2] and AdS/CFT approach to light-like 'cosmologies' [13]. Intersecting brane configurations with localized near-horizon gravitational waves are particularly important in relation to [2]. Even apart from any immediate connections to the contemporary research topics, we believe exact non-linear solutions to be of interest in their own right, as they give us a glimpse of the inner workings of classical higher-dimensional gravities that is impossible to obtain by other means.

The paper consists of two main parts. First, we give a general consideration of an assortment of intersecting black branes with gravitational waves propagating along them. While doing so, we develop a rather systematic treatment of the equations which displays their qualitative structure more explicitly than the previously published investigations. Thereafter, we give a more thorough analysis of a sub-class of such intersecting brane solutions, for which constraints are imposed on the way the branes intersect so that the equations of motion become solvable.

2 General considerations

The low-energy effective action for the supergravity system coupled to dilaton and n_A-form field strengths is given by

$$I = \frac{1}{16\pi G_D} \int d^D x \sqrt{-g} \left[R - \frac{1}{2}(\partial\Phi)^2 - \sum_{A=1}^{m} \frac{1}{2n_A!} e^{a_A \Phi} F_{n_A}^2 \right],$$

(2)

where G_D is the Newton constant in D dimensions and g is the determinant of the metric. The last term includes both RR and NS-NS field strengths, and $a_A = \frac{1}{2}(5 - n_A)$ for RR field strength and $a_A = -1$ for NS-NS 3-form. In the eleven-dimensional supergravity, there is a four-form and no dilaton. We set the fermions and other background fields to zero.

2.1 Metric and field equations

From the action (2), one can derive the field equations

$$R_{\mu\nu} = \frac{1}{2} \partial_\mu \Phi \partial_\nu \Phi + \sum_A \frac{1}{2n_A!} e^{a_A \Phi} \left[n_A \left(F_{n_A}^2 \right)_{\mu\nu} - \frac{n_A - 1}{D - 2} F_{n_A}^2 g_{\mu\nu} \right],$$

(3)

$$\Box \Phi = \sum_A \frac{a_A}{2n_A!} e^{a_A \Phi} F_{n_A}^2,$$

(4)

$$\partial_{\mu_1}\left(\sqrt{-g}\,e^{a_A\Phi}F^{\mu_1\cdots\mu_{n_A}}\right)=0\,,\tag{5}$$

where $F_{n_A}^2$ denotes $F_{\mu_1\cdots\mu_{n_A}}F^{\mu_1\cdots\mu_{n_A}}$ and $(F_{n_A}^2)_{\mu\nu}$ denotes $F_{\mu\rho\cdots\sigma}F_\nu{}^{\rho\cdots\sigma}$.

The Bianchi identity for the form field is given by

$$\partial_{[\mu}F_{\mu_1\cdots\mu_{n_A}]}=0.\tag{6}$$

In this paper we assume the following metric form:

$$\begin{aligned}
ds_D^2 &= e^{2\Xi(u,r)}\left[-2dudv+K(u,y^\alpha,r)du^2\right]+\sum_{\alpha=1}^{d-2}e^{2Z_\alpha(u,r)}(dy^\alpha)^2\\
&\quad+e^{2B(u,r)}\left(dr^2+r^2d\Omega_{\tilde d+1}^2\right),
\end{aligned}\tag{7}$$

where $D=d+\tilde d+2$, the coordinates u, v and $y^\alpha,(\alpha=1,\ldots,d-2)$ parameterize the d-dimensional worldvolume where the branes belong, and the remaining $\tilde d+2$ coordinates r and angles are transverse to the brane worldvolume, $d\Omega_{\tilde d+1}^2$ is the line element of the $(\tilde d+1)$-dimensional sphere. (Note that u and v are null coordinates.) The metric components Ξ,Z_α,B and the dilaton Φ are assumed to be functions of u and r, whereas K depends on u,y^α and r. We shall denote derivatives with respect to u and r by dot and prime, respectively. For the field strength backgrounds, we take

$$F_{n_A}=E_A'(u,r)\,du\wedge dv\wedge dy^{\alpha_1}\wedge\cdots\wedge dy^{\alpha_{q_A-1}}\wedge dr,\tag{8}$$

where $n_A=q_A+2$. With our ansatz, the Einstein equations (3) reduce to

$$\begin{aligned}
&\sum_{\alpha=1}^{d-2}\ddot Z_\alpha+(\tilde d+2)\ddot B+\sum_{\alpha=1}^{d-2}\dot Z_\alpha^2+(\tilde d+2)\dot B^2-2\dot\Xi\left[\sum_{\alpha=1}^{d-2}\dot Z_\alpha+(\tilde d+2)\dot B\right]\\
&+\frac12\sum_{\alpha=1}^{d-2}e^{2(\Xi-Z_\alpha)}\partial_\alpha^2 K+e^{2(\Xi-B)}\left[K\Xi''+\frac12 K''+\left(\Xi'K+\frac12 K'\right)\left(U'+\frac{\tilde d+1}{r}\right)\right]\\
&\qquad\qquad=\sum_A\frac{D-q_A-3}{2(D-2)}e^{2(\Xi-B)}KS_A(E_A')^2-\frac12(\dot\Phi)^2,
\end{aligned}\tag{9}$$

$$\begin{aligned}
&\dot\Xi'+\sum_{\alpha=1}^{d-2}\dot Z_\alpha'+(\tilde d+1)\dot B'-\left[\sum_{\alpha=1}^{d-2}\dot Z_\alpha+(\tilde d+2)\dot B\right]\Xi'-\dot B\sum_{\alpha=1}^{d-2}Z_\alpha'+\sum_{\alpha=1}^{d-2}\dot Z_\alpha Z_\alpha'\\
&\qquad\qquad+\frac12\dot\Phi\Phi'=0,
\end{aligned}\tag{10}$$

$$\Xi''+\left(U'+\frac{\tilde d+1}{r}\right)\Xi'=\sum_A\frac{D-q_A-3}{2(D-2)}S_A(E_A')^2,\tag{11}$$

$$Z_\alpha''+\left(U'+\frac{\tilde d+1}{r}\right)Z_\alpha'=\sum_A\frac{\delta_A^{(\alpha)}}{2(D-2)}S_A(E_A')^2,\tag{12}$$

$$U''+B''-\left(2\Xi'+\sum_{\alpha=1}^{d-2}Z_\alpha'-\frac{\tilde d+1}{r}\right)B'+2(\Xi')^2+\sum_{\alpha=1}^{d-2}(Z_\alpha')^2$$

3

$$= -\frac{1}{2}(\Phi')^2 + \sum_A \frac{D - q_A - 3}{2(D-2)} S_A (E_A')^2, \tag{13}$$

$$B'' + \left(U' + \frac{\tilde{d}+1}{r}\right) B' + \frac{U'}{r} = -\sum_A \frac{q_A + 1}{2(D-2)} S_A (E_A')^2, \tag{14}$$

where U, S_A and $\delta_A^{(\alpha)}$ are defined by

$$U \equiv 2\Xi + \sum_{\alpha=1}^{d-2} Z_\alpha + \tilde{d} B, \tag{15}$$

$$S_A \equiv \exp\left[\epsilon_A a_A \Phi - 2\left(2\Xi + \sum_{\alpha \in q_A} Z_\alpha\right)\right], \tag{16}$$

and

$$\delta_A^{(\alpha)} = \begin{cases} D - q_A - 3 \\ -(q_A + 1) \end{cases} \quad \text{for} \quad \begin{cases} y^\alpha \text{ belonging to } q_A\text{-brane} \\ \text{otherwise} \end{cases}, \tag{17}$$

respectively, and $\epsilon_A = +1(-1)$ is for electric (magnetic) backgrounds. The sum of α in Eq. (16) runs over the q_A-brane components in the $(d-2)$-dimensional y^α-space, for example

$$\sum_{\alpha \in q_A} Z_\alpha = \sum_{\alpha_A=1}^{q_A-1} Z_{\alpha_A}. \tag{18}$$

(9)–(14) are the $uu, ur, uv, \alpha\beta, rr$ and ab components of the Einstein equations (3), respectively. The dilaton equation (4) and the equations for the form field (5) and (6) yield

$$e^{-U} r^{-(\tilde{d}+1)} (e^U r^{\tilde{d}+1} \Phi')' = -\frac{1}{2} \sum_A \epsilon_A a_A S_A (E_A')^2, \tag{19}$$

$$\left(r^{\tilde{d}+1} e^U S_A E_A'\right)' = \left(r^{\tilde{d}+1} e^U S_A E_A'\right)^{\cdot} = 0. \tag{20}$$

The configuration implicit in our ansatz is an assortment of intersecting black q_A-branes, each of which is extended in a subset of y^α-directions. Furthermore, each brane is smeared over all the y^α-directions except for those already aligned with its worldvolume. This arrangement ensures translational symmetry in y^α.

The equations of motion we have presented possess a fairly special structure [14] which enables their thorough analysis. Equations (11-14) and (19-20) are exactly identical to those for a u-independent problem (static intersecting black branes). We shall show that these equations reduce to a generalized Toda system (in special cases, such Toda-like equations can be explicitly integrated) with one constraint on the total 'energy' (i.e., the value of the Toda Hamiltonian). In our context, which includes u-dependence, all the integration constants of the equations without u-derivatives should be understood as

4

functions of u. These functions of u correspond to the amplitudes of the various non-linear gravitational waves present in our solutions. Substitution of solutions to (11-14) and (19-20) into (10) leads to constraints on the non-linear wave amplitudes and their u-derivatives. We shall prove below that there is always only one such constraint. Finally, equation (9) determines the uu-component of the metric, without introducing any further constraints on the previously obtained solution of (10-14) and (19-20).

2.2 Generalized Toda equations

We shall now analyze explicitly equations (11-14) and (19-20) and show that they reduce to a Toda-like system.

From (20), we learn that

$$Q^{-1}S_A E'_A = c_A \tag{21}$$

is a constant, where $Q^{-1} = r^{\tilde{d}+1}e^U$. Equation (19) tells us that

$$Q^{-1}\Phi' = -\frac{1}{2}\sum_A \epsilon_A a_A c_A E_A + f_\Phi. \tag{22}$$

Similarly, we have

$$Q^{-1}\Xi' = \sum_A \frac{D - q_A - 3}{2(D-2)}c_A E_A + f_\Xi, \tag{23}$$

and

$$Q^{-1}Z'_\alpha = \sum_A \frac{\delta_A^{(\alpha)}}{2(D-2)}c_A E_A + f_\alpha. \tag{24}$$

where f_Φ, f_Ξ and f_α are functions of u. From (11), (12) and (14), we get

$$\left[Q^{-1}U'\right]' + Q^{-1}\frac{\tilde{d}U'}{r} = \sum_A \left\{\left[2(D-q_A-3) + \left(\sum_{\alpha=1}^{d-2}\delta_A^{(\alpha)}\right) - \tilde{d}(q_A+1)\right]\frac{c_A E'_A}{2(D-2)}\right\}$$
$$= 0, \tag{25}$$

where we have used the relation

$$\sum_{\alpha=1}^{d-2}\delta_A^{(\alpha)} = \tilde{d}(q_A+1) - 2(D - q_A - 3). \tag{26}$$

So, from Eq. (25), we have

$$\left(r^{2\tilde{d}+1}e^U U'\right)' = 0, \tag{27}$$

and thus

$$e^U = h_U - \frac{f_U}{(2\tilde{d})r^{2\tilde{d}}}, \tag{28}$$

where f_U and h_U are functions of u.

We now show that (21)-(24) can be re-written as a generalized Toda system. The appearance of a Toda structure is not surprising in our context, since a 2×2 Toda-like system has already been observed for dyonic (static) solutions [25]. In our case, there is an arbitrary number of different form charges, and the number of Toda equations will equal the number of charges, just like in [25].

Introducing a new variable w by

$$\frac{d}{dw} = Q^{-1}\frac{d}{dr}, \tag{29}$$

we find that (22)–(24) can be integrated as

$$
\begin{aligned}
\Phi &= -\frac{1}{2}\sum_A \epsilon_A a_A c_A F_A + f_\Phi(u)w + h_\Phi(u), \\
\Xi &= \sum_A \frac{D - q_A - 3}{2(D-2)}c_A F_A + f_\Xi(u)w + h_\Xi(u), \\
Z_\alpha &= \sum_A \frac{\delta_A^{(\alpha)}}{2(D-2)}c_A F_A + f_\alpha(u)w + h_\alpha(u),
\end{aligned} \tag{30}
$$

where we have defined

$$F_A \equiv \int dw E_A. \tag{31}$$

Substituting these into (21), we obtain

$$\frac{d^2 F_A}{dw^2} = c_A \exp\left[\sum_B 2c_A^{-1}M_{AB}^{(2)}F_B + 2M_A^{(1)}c_A^{-1}w + \left(4h_\Xi + 2\sum_{\alpha\in q_A} h_\alpha - \epsilon_A a_A h_\Phi\right)\right], \tag{32}$$

where

$$M_{AB}^{(2)} = \frac{1}{2}\left\{\frac{1}{2}\epsilon_A a_A \epsilon_B a_B + 2\frac{D - q_B - 3}{D-2} + \sum_{\alpha\in q_A}\frac{\delta_B^{(\alpha)}}{D-2}\right\}c_A c_B, \tag{33}$$

$$M_A^{(1)} = \left(2f_\Xi + \sum_{\alpha\in q_A} f_\alpha - \frac{1}{2}\epsilon_A a_A f_\Phi\right)c_A. \tag{34}$$

$M_{AB}^{(2)}$ is actually a symmetric matrix, as shown in Appendix A. The terms linear in w and independent of w in the exponent on the right-hand side of (32) can be absorbed into a redefinition of F_A. Indeed, introducing

$$\tilde{F}_A = F_A + M_{AB}^{(2)}{}^{-1}M_B^{(1)}w + M_{AB}^{(2)}{}^{-1}c_B\left(2h_\Xi + \sum_{\alpha\in q_B} h_\alpha - \frac{1}{2}\epsilon_B a_B h_\Phi\right) \tag{35}$$

(where summation over B is understood), we obtain an explicitly Toda-like system

$$\frac{d^2\tilde{F}_A}{dw^2} = c_A \exp\left[\sum_B 2c_A^{-1}M_{AB}^{(2)}\tilde{F}_B\right]. \tag{36}$$

6

These equations can be derived (as $\partial H/\partial p_A = d\tilde{F}_A/dw$ and $\partial H/\partial \tilde{F}_A = -dp_A/dw$) from the following Toda Hamiltonian:

$$H = \frac{1}{2}\sum_{A,B} M^{(2)\,-1}_{AB} p_A p_B - \frac{1}{2}\sum_A c_A^2 \exp\left[\sum_B 2c_A^{-1}M^{(2)}_{AB}\tilde{F}_B\right]. \tag{37}$$

Generalized Toda equations have been considered extensively in mathematical literature. They are known to be integrable [26, 27] for special choices of $M^{(2)}_{AB}$ (constructed from roots of Lie algebras). The original Toda system (one-dimensional non-linear 'crystal' with exponential interactions between atoms) corresponds to a sparse $M^{(2)}$-matrix ($M^{(2)}_{AB} = 0$ unless $A = B \pm 1$). If $M^{(2)}$ is diagonal, the system splits into independent Liouville equations and is particularly easily integrated. We shall analyze this case explicitly in section 3. Any special form of the $M^{(2)}$-matrix imposes constraints on the arrangement of the intersecting branes.

We have presently analyzed all the equations of motion without u-derivatives, except for (13). We shall now show that (13) can be re-written as a first order equation, which merely imposes one constraint on the integration constants of the equations we have already considered.

Eliminating second derivatives from (13) with the use of (14) and (27) and expressing B through U, Ξ and Z_α according to (15), we get

$$-(\tilde{d}+1)U'^2 - 2\tilde{d}(\tilde{d}+1)\frac{U'}{r} + 2(\tilde{d}+2)(\Xi')^2 + 4\Xi'\sum_\alpha Z'_\alpha + \tilde{d}\sum_\alpha (Z'_\alpha)^2$$

$$+ \sum_{\alpha,\beta} Z'_\alpha Z'_\beta + \frac{\tilde{d}}{2}(\Phi')^2 = \frac{\tilde{d}}{2}\sum_A Qc_A E'_A. \tag{38}$$

From the expression of U given by (28), we find

$$-(\tilde{d}+1)U'^2 - 2\tilde{d}(\tilde{d}+1)\frac{U'}{r} = -2\tilde{d}(\tilde{d}+1)f_U h_U Q^2. \tag{39}$$

Substituting (22)–(24) and (39) into (38), we get

$$\sum_{A,B} M^{(2)}_{AB} E_A E_B + 2\sum_A M^{(1)}_A E_A - \sum_A Q^{-1}c_A E'_A = -M^{(0)}, \tag{40}$$

where

$$\frac{\tilde{d}}{2}M^{(0)} = 2(\tilde{d}+2)f_\Xi^2 + 4\sum_\alpha f_\Xi f_\alpha + \tilde{d}\sum_\alpha f_\alpha^2 + \sum_{\alpha,\beta} f_\alpha f_\beta + \frac{\tilde{d}}{2}f_\Phi^2 - 2\tilde{d}(\tilde{d}+1)f_U h_U. \tag{41}$$

Here we have also used (26) and the fact that

$$\frac{1}{D-2}\sum_\alpha f_\alpha\left(\delta_A^{(\alpha)} + q_A + 1\right) = \sum_{\alpha \in q_A} f_\alpha. \tag{42}$$

7

Re-expressing E_A through $\tilde{F}_A$ according to (31) and (35), and passing to differentiation with respect to w, we discover that (up to w-independent terms) the left-hand side of (40) is nothing but the 'energy' of the Toda system (36), i.e., the Toda Hamiltonian (37) expressed through $\tilde{F}_A$ and their w-derivatives. Hence, by the Toda 'energy' conservation, the left-hand side of (40) cannot depend on w, and it simply imposes one constraint on the integration constants of the Toda system (36).

To summarize, we have reduced equations (11-14) and (19-20) to a generalized Toda system. Such equations have been extensively studied and they can be integrated in special cases (corresponding to special arrangements of intersecting branes in our context). There is a number of arbitrary functions of u appearing in the solution (these functions of u are integration constants of the differential equations containing only derivatives with respect to r): f_U, h_U, f_Φ, f_Ξ, f_α, h_Φ, h_Ξ, h_α and $2N$ more functions[1] arising from a general solution to the Toda system (where N is the number of form charges). There is one algebraic constraint on these functions of u due to equation (40).

2.3 The dot-prime equation

Up to this point, we have shown that equations (11-14) and (19-20) fix all the r-dependences in the metric (except for the uu-component), dilaton and the form fields, and leave a large number of arbitrary functions of u (the non-linear gravitational wave amplitudes), which we shall here symbolically denote as $\eta_i(u)$ (the index i runs over all the arbitrary functions of u introduced in the preceding derivations). In other words, after equations (11-14) and (19-20) have been solved, we obtain $B = B(r; \eta_i(u))$ (where the r-dependence is completely determined) and similar expressions for the other functions appearing in our ansatz for the metric, the dilaton and the form fields.

We now have to substitute the solutions to (11-14) and (19-20) into the 'dot-prime' equation (10). Generically, since (10) only contains first derivatives with respect to u, this should result in the following structure:

$$\mathcal{G}(r; \eta_i(u), \dot{\eta}_i(u)) = 0, \tag{43}$$

where the r-dependence is completely fixed due to equations (11-14) and (19-20). In principle, (43) may contain an infinite number of constraints (one for each value of r) on the finite number of functions $\eta_i(u)$, leaving no solutions. We shall now show that, due to an interplay between the structure of (10) on the one hand and (11-14) and (19-20) on the other hand, this does not happen, and (10) in fact always enforces only one constraint on $\eta_i(u)$ in the form of a first-order differential equation with respect to u:

$$\mathcal{G}(\eta_i(u), \dot{\eta}_i(u)) = 0. \tag{44}$$

(This structure has been observed in all the previously derived solutions of this sort, e.g., [1, 14, 15], and we now infer it from the equations of motion in a general fashion, without relying on explicit functional form of the solutions, which is how it was seen in the previously published considerations.) To this end, we assume that (11-14) and (19-20)

---

[1]A certain number of integration constants of the Toda system may be absorbed into a re-definition of h_Φ, h_Ξ and h_α.

are satisfied and examine the left-hand side of (10). Differentiating the said left-hand side with respect to r, and eliminating second derivatives by use of (11), (12) and (14), we obtain

$$-\left(U' + \frac{\tilde{d}+1}{r}\right)\left[\dot{\Xi}' + \sum_{\alpha=1}^{d-2}\dot{Z}'_\alpha + (\tilde{d}+1)\dot{B}' - \left\{\sum_{\alpha=1}^{d-2}\dot{Z}_\alpha + (\tilde{d}+1)\dot{B}\right\}\Xi'\right.$$

$$\left. -\dot{B}\sum_{\alpha=1}^{d-2}Z'_\alpha + \sum_{\alpha=1}^{d-2}\dot{Z}_\alpha Z'_\alpha + \frac{1}{2}\dot{\Phi}\Phi'\right] + R_1 + R_2, \tag{45}$$

where

$$R_1 = -\dot{U}'\left(\Xi' + \sum_{\alpha=1}^{d-2}Z'_\alpha\right) - (\tilde{d}+1)\left(\dot{U}'B' + \frac{\dot{U}'}{r}\right) - \left(\sum_{\alpha=1}^{d-2}\dot{Z}'_\alpha + (\tilde{d}+2)\dot{B}'\right)\Xi'$$

$$-\dot{B}'\sum_{\alpha=1}^{d-2}Z'_\alpha + \sum_{\alpha=1}^{d-2}\dot{Z}'_\alpha Z'_\alpha + \frac{1}{2}\dot{\Phi}'\Phi'$$

$$R_2 = \sum_A \frac{D-q_A-3}{2(D-2)}(S_A E'^2_A)^\cdot + \sum_A\sum_{\alpha=1}^{d-2}\frac{\delta_A^{(\alpha)}}{2(D-2)}(S_A E'^2_A)^\cdot - (\tilde{d}+1)\sum_A \frac{q_A+1}{2(D-2)}(S_A E'^2_A)^\cdot$$

$$-\left[\sum_{\alpha=1}^{d-2}\dot{Z}_\alpha + (\tilde{d}+2)\dot{B}\right]\sum_A\frac{D-q_A-3}{2(D-2)}S_A E'^2_A - \dot{B}\sum_A\sum_{\alpha=1}^{d-2}\frac{\delta_A^{(\alpha)}}{2(D-2)}S_A E'^2_A$$

$$+\sum_A\sum_{\alpha=1}^{d-2}\dot{Z}_A\frac{\delta_A^{(\alpha)}}{2(D-2)}S_A E'^2_A - \frac{1}{4}\dot{\Phi}\sum_A\epsilon_A a_A S_A E'^2_A \tag{46}$$

Eliminating B in favor of U from R_1, we obtain

$$R_1 = \left[-\frac{\tilde{d}+1}{2\tilde{d}}U'^2 - (\tilde{d}+1)\frac{U'}{r} + \frac{2}{\tilde{d}}\Xi'\sum_{\alpha=1}^{d-2}Z'_\alpha + \frac{\tilde{d}+2}{\tilde{d}}\Xi'^2\right.$$

$$\left. +\frac{1}{2\tilde{d}}\sum_{\alpha,\beta}Z'_\alpha Z'_\beta + \frac{1}{2}\sum_{\alpha=1}^{d-2}Z'^2_\alpha + \frac{1}{4}\Phi'^2\right]^\cdot. \tag{47}$$

On the other hand, with the help of (26) and (42), R_2 can be reduced to

$$R_2 = -\frac{1}{2}\sum_A\left[(S_A E'^2_A)^\cdot + S_A E'^2_A\left(\dot{U} + \frac{\dot{S}_A}{2S_A}\right)\right]$$

$$= -\frac{1}{4}\sum_A(c_A Q E'_A)^\cdot, \tag{48}$$

where we have also used (20) and (21) in deriving the second equality. Consequently $R_1 + R_2$ vanishes by (38).

From this analysis, we conclude that, if (11-14) and (19-20) are satisfied,

$$\left[\dot{\Xi}' + \sum_{\alpha=1}^{d-2}\dot{Z}'_\alpha + (\tilde{d}+1)\dot{B}' - \left\{\sum_{\alpha=1}^{d-2}\dot{Z}_\alpha + (\tilde{d}+1)\dot{B}\right\}\Xi' - \dot{B}\sum_{\alpha=1}^{d-2}Z'_\alpha + \sum_{\alpha=1}^{d-2}\dot{Z}_\alpha Z'_\alpha + \frac{1}{2}\dot{\Phi}\Phi'\right]'$$

9

$$= -\left(U' + \frac{\tilde{d}+1}{r}\right)\left[\dot{\Xi}' + \sum_{\alpha=1}^{d-2}\dot{Z}'_\alpha + (\tilde{d}+1)\dot{B}' - \left\{\sum_{\alpha=1}^{d-2}\dot{Z}_\alpha + (\tilde{d}+1)\dot{B}\right\}\Xi' - \dot{B}\sum_{\alpha=1}^{d-2}Z'_\alpha\right.$$
$$\left. + \sum_{\alpha=1}^{d-2}\dot{Z}_\alpha Z'_\alpha + \frac{1}{2}\dot{\Phi}\Phi'\right]. \tag{49}$$

Hence, the left-hand side of (10) has the form

$$\dot{\Xi}' + \sum_{\alpha=1}^{d-2}\dot{Z}'_\alpha + (\tilde{d}+1)\dot{B}' - \left\{\sum_{\alpha=1}^{d-2}\dot{Z}_\alpha + (\tilde{d}+1)\dot{B}\right\}\Xi' - \dot{B}\sum_{\alpha=1}^{d-2}Z'_\alpha$$
$$+ \sum_{\alpha=1}^{d-2}\dot{Z}_\alpha Z'_\alpha + \frac{1}{2}\dot{\Phi}\Phi' = \frac{\mathcal{F}(u)}{r^{\tilde{d}+1}e^U}. \tag{50}$$

Since (50) only contains first derivatives with respect to u, and since u only enters the metric and the dilaton through $\eta_i(u)$, $\mathcal{F}(u)$ can be expressed through $\eta_i(u)$ and their first derivatives, i.e., $\mathcal{F}(u) = \mathcal{G}(\eta_i(u), \dot{\eta}_i(u))$. Hence, (10) takes the form (44), as previously claimed.

2.4 The K-equation and non-linear wave counting

By use of (11), (9) becomes

$$\sum_{\alpha=1}^{d-2}\left[\ddot{Z}_\alpha - 2\dot{\Xi}\dot{Z}_\alpha + \dot{Z}_\alpha^2\right] + (\tilde{d}+2)\left[\ddot{B} - 2\dot{\Xi}\dot{B} + \dot{B}^2\right] + \frac{1}{2}(\dot{\Phi})^2$$
$$+ \frac{1}{2}\sum_{\alpha=1}^{d-2}e^{2(\Xi-Z_\alpha)}\partial_\alpha^2 K + \frac{1}{2}e^{2(\Xi-B)}Q\left(\frac{K'}{Q}\right)' = 0. \tag{51}$$

This equation determines K (and hence the uu-component of the metric) after equations (10-14) and (19-20), which do not contain K, have been solved.

The analysis of this equation essentially does not differ from the single black brane case considered in [1]. It is convenient to enforce the Brinkmann parametrization (1) for the plane wave at $r \to \infty$. Then Ξ, B and Z_α go to 0 for large r. For the asymptotic large r plane wave, it is not difficult to include the $\alpha\beta$ polarizations by assuming

$$K(u, r, y^\alpha) = k(u, r) + K_{\alpha\beta}(u)y^\alpha y^\beta. \tag{52}$$

Then, (51) reduces to

$$\sum_{\alpha=1}^{d-2}\left[\ddot{Z}_\alpha - 2\dot{\Xi}\dot{Z}_\alpha + \dot{Z}_\alpha^2\right] + (\tilde{d}+2)\left[\ddot{B} - 2\dot{\Xi}\dot{B} + \dot{B}^2\right] + \frac{1}{2}(\dot{\Phi})^2$$
$$+ \frac{1}{2}\sum_{\alpha=1}^{d-2}e^{2(\Xi-Z_\alpha)}K_{\alpha\alpha} + \frac{1}{2}e^{2(\Xi-B)}Q\left(\frac{k'}{Q}\right)' = 0. \tag{53}$$

10

One then simply takes a solution for Ξ, B, Z_α and Φ (with all the arbitrary functions of u contained therein), specifies $K_{\alpha\beta}(u)$ and solves the above equation for $k(u,r)$. The arbitrary functions of u contained in our solution are thus $K_{\alpha\beta}(u)$ and the functions of u contained in the solution of (10-14) and (19-20) subject to Ξ, B and Z_α going to 0 at large r.

If Ξ, B, Z_α and Φ are known, determining k always merely amounts to two integrations over r. We have thus shown that the original system of partial differential equations (9-14) and (19-20) with respect to r and u reduces to:

- a generalized Toda system (36) of ordinary differential equations with respect to r, which has been extensively studied and admits analytic solutions in special cases;

- an ordinary differential equation (44) with respect to u;

- ordinary integrations over r needed to solve (53).

This is an enormous simplification, and one is left with a clear picture of the kind of non-linear gravitational waves propagating in our system. In the subsequent section, we give a more explicit analysis to one of the special cases when the Toda system can be integrated.

3   A class of explicit solutions

The Toda system (36) can obviously be integrated when $M^{(2)}$ is diagonal. (In that case, the Toda system splits into individual Liouville equations, each of which can be solved.) Imposing the vanishing condition on the off-diagonal components of $M^{(2)}$ introduces an intersection rule constraining the arrangement of the branes [28, 29].

To simplify the discussion, let us change the notation as follows:

$$M^{(2)}_{AA} = \xi_A c^2_A, \qquad M^{(1)}_A = \kappa_A c_A, \tag{54}$$

where

$$\xi_A = \frac{(q_A + 1)(D - q_A - 3)}{2(D-2)} + \frac{1}{4}a^2_A = 1, \tag{55}$$

$$\kappa_A = 2f_\Xi + \sum_{\alpha \in q_A} f_\alpha - \frac{1}{2}\epsilon_A a_A f_\Phi. \tag{56}$$

The second equality in (55) holds for all supergravities including those corresponding to ten-dimensional superstrings and eleven-dimensional M-theory. With these specifications, the Toda equations (36) become

$$\frac{d^2\tilde{F}_A}{dw^2} = c_A e^{2c_A \tilde{F}_A}, \tag{57}$$

which is solved by

$$\tilde{F}_A = -c^{-1}_A \ln\left(\frac{\sinh\left(c_A \sigma_A(w - \omega_A(u))\right)}{\sigma_A}\right), \tag{58}$$

where $\sigma_A(u)$ and $\omega_A(u)$ are integration constants of the Liouville equations (57).

From (35) and (54),

$$F_A = \tilde{F}_A - c_A^{-1}(\kappa_A w + N_A),\tag{59}$$

where we have defined

$$N_A = 2h_\Xi + \sum_{\alpha \in q_A} h_\alpha - \frac{1}{2}\epsilon_A a_A h_\Phi \,.\tag{60}$$

Considering (40), we find

$$-M^{(0)} = \sum_A \left(c_A^2 \left(\frac{dF_A}{dw}\right)^2 + 2c_A\kappa_A\frac{dF_A}{dw} - c_A\frac{d^2F_A}{dw^2} \right)$$
$$= \sum_A \left(c_A^2 \left(\frac{d\tilde{F}_A}{dw}\right)^2 - \kappa_A^2 - c_A^2 e^{2c_A\tilde{F}_A} \right) = \sum_A \left(c_A^2\sigma_A^2 - \kappa_A^2 \right),\tag{61}$$

which gives one algebraic constraint on the integration constants of the Toda system, as it should per our general discussion in section 2. One can recognize the energy of (57) among the expressions in (61).

We now turn to the 'dot-prime' equation (10). Eliminating B from (10) by the use of (15), we obtain

$$-\frac{\tilde{d}+2}{\tilde{d}}(\dot{\Xi}' + \dot{U}\Xi') - \frac{1}{\tilde{d}}\left(\sum_\alpha \dot{Z}'_\alpha + \dot{U}\sum_\alpha Z'_\alpha \right) + \frac{\tilde{d}+1}{\tilde{d}}\dot{U}' + \frac{1}{2}\dot{\Phi}\Phi'$$
$$+\frac{2}{\tilde{d}}\Xi'\sum_\alpha \dot{Z}_\alpha + \frac{2}{\tilde{d}}\dot{\Xi}\sum_\alpha Z'_\alpha + 2\frac{\tilde{d}+2}{\tilde{d}}\dot{\Xi}\Xi' + \sum_\alpha \dot{Z}_\alpha Z'_\alpha + \frac{1}{\tilde{d}}\sum_\alpha Z'_\alpha \sum_\beta \dot{Z}_\beta = 0\,.\tag{62}$$

Considering

$$\dot{\Xi}' + \dot{U}\Xi' = Q(Q^{-1}\Xi')^{\cdot},\qquad \sum_\alpha \dot{Z}'_\alpha + \dot{U}\sum_\alpha Z'_\alpha = Q\left(Q^{-1}\sum_\alpha Z'_\alpha\right)^{\cdot},\tag{63}$$

and (22)–(24), (26) and (42), we have

$$-\frac{\tilde{d}+2}{\tilde{d}}\dot{f}_\Xi - \frac{1}{\tilde{d}}\sum_\alpha \dot{f}_\alpha + \frac{\tilde{d}+1}{\tilde{d}}Q^{-1}\dot{U}' - \frac{1}{2}\sum_A c_A\dot{E}_A$$
$$+\frac{1}{2}\sum_A \left(2\dot{\Xi} + \sum_{\alpha \in q_A} \dot{Z}_\alpha - \frac{1}{2}\epsilon_A a_A\dot{\Phi} \right)c_A E_A + \frac{2}{\tilde{d}}\sum_\alpha f_\alpha\dot{\Xi} + 2\frac{\tilde{d}+2}{\tilde{d}}f_\Xi\dot{\Xi}$$
$$+\frac{2}{\tilde{d}}f_\Xi\sum_\alpha \dot{Z}_\alpha + \sum_\alpha f_\alpha\dot{Z}_\alpha + \frac{1}{\tilde{d}}\sum_\beta f_\beta\sum_\alpha \dot{Z}_\alpha + \frac{1}{2}f_\Phi\dot{\Phi} = 0\,.\tag{64}$$

Due to (26), (30) and (54), we get

$$2\dot{\Xi} + \sum_{\alpha \in q_A} \dot{Z}_\alpha - \frac{1}{2}\epsilon_A a_A\dot{\Phi} = \dot{N}_A + \dot{\kappa}_A w + \kappa_A\dot{w} + c_A\dot{F}_A\,.\tag{65}$$

12

Further applications of (26), (30) and (42) yield

$$
\frac{2}{\tilde{d}}\sum_\alpha f_\alpha\dot{\Xi} + 2\frac{\tilde{d}+2}{\tilde{d}}f_\Xi\dot{\Xi} + \frac{2}{\tilde{d}}f_\Xi\sum_\alpha \dot{Z}_\alpha + \sum_\alpha f_\alpha\dot{Z}_\alpha + \frac{1}{\tilde{d}}\sum_\beta f_\beta\sum_\alpha \dot{Z}_\alpha + \frac{1}{2}f_\Phi\dot{\Phi}
$$

$$
= \frac{2}{\tilde{d}}\sum_\alpha f_\alpha\dot{h}_\Xi + 2\frac{\tilde{d}+2}{\tilde{d}}f_\Xi\dot{h}_\Xi + \frac{2}{\tilde{d}}f_\Xi\sum_\alpha \dot{h}_\alpha + \sum_\alpha f_\alpha\dot{h}_\alpha + \frac{1}{\tilde{d}}\sum_\beta f_\beta\sum_\alpha \dot{h}_\alpha + \frac{1}{2}f_\Phi\dot{h}_\Phi
$$

$$
+ 2(\tilde{d}+1)(f_U h_U)\dot{w} + (\tilde{d}+1)(f_U h_U)^\cdot w + \frac{1}{2}M^{(0)}\dot{w} + \frac{1}{4}\dot{M}^{(0)}w + \frac{1}{2}\sum_A c_A\kappa_A\dot{F}_A \,.
$$

$$(66)$$

So the dot-prime equation becomes

$$
M + (\tilde{d}+1)Q^{-1}\dot{U}' + 2\tilde{d}(\tilde{d}+1)(f_U h_U)\dot{w} + \tilde{d}(\tilde{d}+1)(f_U h_U)^\cdot w
$$

$$
+ \frac{\tilde{d}}{2}\sum_A\left[-c_A\dot{E}_A + \left(\dot{N}_A + (\kappa_A w)^\cdot + c_A\dot{F}_A\right)c_A E_A\right]
$$

$$
+ \frac{\tilde{d}}{2}M^{(0)}\dot{w} + \frac{\tilde{d}}{4}\dot{M}^{(0)}w + \frac{\tilde{d}}{2}\sum_A c_A\kappa_A\dot{F}_A = 0 \,,
$$

$$(67)$$

where

$$
M \equiv -(\tilde{d}+2)\dot{f}_\Xi - \sum_\alpha \dot{f}_\alpha + 2\sum_\alpha f_\alpha\dot{h}_\Xi + 2(\tilde{d}+2)f_\Xi\dot{h}_\Xi
$$

$$
+ 2f_\Xi\sum_\alpha \dot{h}_\alpha + \tilde{d}\sum_\alpha f_\alpha\dot{h}_\alpha + \sum_\beta f_\beta\sum_\alpha \dot{h}_\alpha + \frac{\tilde{d}}{2}f_\Phi\dot{h}_\Phi \,.
$$

$$(68)$$

It is easy to deduce that

$$
Q^{-1}\dot{U}' + 2\tilde{d}(f_U h_U)\dot{w} + \tilde{d}(f_U h_U)^\cdot w = 0 \,,
$$

$$(69)$$

and we get

$$
M + \frac{\tilde{d}}{2}\sum_A c_A\left\{-\dot{E}_A + \left(\dot{N}_A + (\kappa_A w)^\cdot + c_A\dot{F}_A\right)E_A + \kappa_A\dot{F}_A\right\} + \frac{\tilde{d}}{2}M^{(0)}\dot{w} + \frac{\tilde{d}}{4}\dot{M}^{(0)}w = 0 \,, \quad (70)
$$

or, using (59),

$$
M + \frac{\tilde{d}}{2}\sum_A c_A\left\{c_A\dot{\tilde{F}}_A\left(\frac{d\tilde{F}_A}{dw}\right) - \left(\frac{d\tilde{F}_A}{dw}\right)^\cdot + c_A^{-1}\left(\dot{\kappa}_A - \kappa_A(\kappa_A w)^\cdot - \kappa_A\dot{N}_A\right)\right\}
$$

$$
+ \frac{\tilde{d}}{2}M^{(0)}\dot{w} + \frac{\tilde{d}}{4}\dot{M}^{(0)}w = 0 \,.
$$

$$(71)$$

From (58),

$$
c_A\dot{\tilde{F}}_A\left(\frac{d\tilde{F}_A}{dw}\right) - \left(\frac{d\tilde{F}_A}{dw}\right)^\cdot = c_A\sigma_A\left(\sigma_A(w - \omega_A)\right)^\cdot \,.
$$

$$(72)$$

13

Hence, we get

$$
\begin{aligned}
M + \frac{\tilde{d}}{2} \sum_A \left\{ \dot{\kappa}_A - \kappa_A \dot{N}_A - c_A^2 \sigma_A (\sigma_A \omega_A)^\cdot \right\} \\
+ \frac{\dot{w}\tilde{d}}{2} \left(\sum_A \left(c_A^2 \sigma_A^2 - \kappa_A^2 \right) + M^{(0)} \right) + \frac{w\tilde{d}}{4} \left(\sum_A \left(c_A^2 \sigma_A^2 - \kappa_A^2 \right) + M^{(0)} \right)^\cdot = 0 \,.
\end{aligned} \tag{73}
$$

The second line vanishes by (61). Hence, the 'dot-prime' equation (10) reduces to a single ordinary differential equation

$$
M + \frac{\tilde{d}}{2} \sum_A \left\{ \dot{\kappa}_A - \kappa_A \dot{N}_A - c_A^2 \sigma_A (\sigma_A \omega_A)^\cdot \right\} = 0, \tag{74}
$$

as per our general discussion in section 2.

4   Conclusions

We have considered intersecting black supergravity branes with strong gravitational waves propagating along their worldvolume. The problem of finding the corresponding supergravity solutions is reduced to ordinary differential equations, including the celebrated generalized Toda system, which can be solved explicitly in special cases.

Methodologically, our treatment presents a few substantial improvements over the previously published material. For the static intersecting black brane case (which is a triviality in our present context, as the gravitational waves are set to zero), our discussion generalizes the treatment of the equations without the assumptions made in [28, 29], where $M^{(2)}$ was taken to be diagonal (a convenient representation for this matrix is given in Appendix A), and displays a connection to generalized Toda systems in the spirit of [25] (see also [30]). For the non-trivial case with finite amplitude gravitational waves turned on, we give a much more thorough and general treatment of the ur-component of Einstein's equations and show, relying only on the equations of motion, but not on their explicit solutions, that it always reduces to a single ordinary differential equation. In the previous publications [1, 14, 15], this fact was demonstrated using the explicit form of solutions. Under our present circumstances, explicit solutions are only available in some special cases, so a general proof of the sort presented here is indispensable to maintain analytic control over the equations of motion. It also gives a much clearer picture of the inner working of the formalism than the derivations of [1, 14, 15].

Physically, it would be interesting to apply our solutions to the study of AdS/CFT realizations of light-like 'cosmologies' (along the lines of [13]) and black hole physics (along the lines of [2]). We defer these subjects to future investigations.

Mathematically, we have seen how very similar algebraic structures enable a thorough analysis of the gravitational equations of motion in a sequence of successively more elaborate examples ([1, 14, 15] and the present publication). It would be interesting to understand the emergence of this structure in a precise and coordinate-independent manner. At this point, it is clear that the presence of a null Killing vector is essential, but it

is not obvious what other assumptions one needs to make about the geometry to create a maximally general setting in which the simplifications (of the sort we have thus far discovered in specialized examples) do occur.

Acknowledgement

A Some properties of the $M^{(2)}$-matrix

In (33), we have defined $M^{(2)}$ by

$$M^{(2)}_{AB} = \frac{1}{2}\left\{ \frac{1}{2}\epsilon_A a_A \epsilon_B a_B + 2\frac{D - q_B - 3}{D - 2} + \sum_{\alpha \in q_A} \frac{\delta_B^{(\alpha)}}{D - 2} \right\} c_A c_B.$$

(75)

We shall now show that $M^{(2)}$ is in fact symmetric by deriving a manifestly symmetric representation of the above formula.

First, from (42), we have

$$\sum_{\alpha \in q_A} \delta_B^{(\alpha)} = \frac{1}{D-2}\sum_\alpha \delta_B^{(\alpha)}\left(\delta_A^{(\alpha)} + q_A + 1 \right) = \sum_\alpha \frac{\delta_B^{(\alpha)}\delta_A^{(\alpha)}}{D-2} + \sum_\alpha \frac{\delta_B^{(\alpha)}}{D-2}(q_A + 1).$$

(76)

Considering (26), we find

$$\sum_{\alpha \in q_A} \delta_B^{(\alpha)} = \sum_\alpha \frac{\delta_B^{(\alpha)}\delta_A^{(\alpha)}}{D-2} + \frac{1}{D-2}(q_A + 1)\left[\tilde{d}(q_B + 1) - 2(D - q_B - 3) \right].$$

(77)

Consequently,

$$M^{(2)}_{AB} = \left\{ \frac{(D - q_A - 3)(D - q_B - 3)}{(D - 2)^2} + \sum_\alpha \frac{\delta_B^{(\alpha)}\delta_A^{(\alpha)}}{2(D - 2)^2} \right.$$
$$\left. + \frac{\tilde{d}(q_A + 1)(q_B + 1)}{2(D - 2)^2} + \frac{1}{4}\epsilon_A a_A \epsilon_B a_B \right\} c_A c_B$$

(78)

The latter expression is manifestly symmetric, which proves our claim.

References

[1] L. M. Cao, O. Evnin and N. Ohta, JHEP **1104** (2011) 105 [arXiv:1101.4455 [hep-th]].

[2] C. G. Callan and J. M. Maldacena, Nucl. Phys. B **472** (1996) 591 [arXiv:hep-th/9602043].

[3] G. C. McVittie, Mon. Not. R. Astron. Soc. **93** (1933) 325; Ap. J. **143** (1966) 682; General Relativity and Cosmology, 2nd Edition, University of Illinois Press 1962.

[4] N. Kaloper, M. Kleban and D. Martin, Phys. Rev. D **81** (2010) 104044 [arXiv:1003.4777 [hep-th]].

[5] H. Nariai , Sci. Rep. Tohoku Univ. **34** (1950) 160; Sci. Rep. Tohoku Univ. **35** (1951) 62.

[6] R. Bousso, arXiv:hep-th/0205177.

[7] K. i. Maeda, N. Ohta and K. Uzawa, JHEP **0906** (2009) 051 [arXiv:0903.5483 [hep-th]].

[8] M. Blau and M. O'Loughlin, JHEP **0809** (2008) 097 [arXiv:0806.3255 [hep-th]].

[9] N. Ohta, K. L. Panigrahi and S. Siwach, Nucl. Phys. B **674** (2003) 306 [Erratum-ibid. B **748** (2006) 333] [arXiv:hep-th/0306186].

[10] R. R. Nayak, K. L. Panigrahi and S. Siwach, Phys. Lett. B **640** (2006) 214 [arXiv:hep-th/0605278].

[11] N. Ohta and K. L. Panigrahi, Phys. Rev. D **74** (2006) 126003 [arXiv:hep-th/0610015].

[12] K. i. Maeda, N. Ohta, M. Tanabe and R. Wakebe, JHEP **0906** (2009) 036 [arXiv:0903.3298 [hep-th]].

[13] S. R. Das, J. Michelson, K. Narayan and S. P. Trivedi, Phys. Rev. D **74** (2006) 026002 [arXiv:hep-th/0602107]; Phys. Rev. D **75** (2007) 026002 [arXiv:hep-th/0610053].

[14] B. Craps, F. De Roo, O. Evnin and F. Galli, JHEP **0907** (2009) 058 [arXiv:0905.1843 [hep-th]].

[15] K. i. Maeda, N. Ohta, M. Tanabe and R. Wakebe, JHEP **1004** (2010) 013 [arXiv:1001.2640 [hep-th]].

[16] E. G. Gimon, A. Hashimoto, V. E. Hubeny, O. Lunin and M. Rangamani, JHEP **0308** (2003) 035 [arXiv:hep-th/0306131].

[17] B. Craps, S. Sethi and E. P. Verlinde, JHEP **0510** (2005) 005 [arXiv:hep-th/0506180].

[18] B. Chen, Phys. Lett. B **632** (2006) 393 [arXiv:hep-th/0508191].

[19] T. Ishino, H. Kodama and N. Ohta, Phys. Lett. B **631** (2005) 68 [arXiv:hep-th/0509173].

[20] D. Robbins and S. Sethi, JHEP **0602** (2006) 052 [arXiv:hep-th/0509204].

[21] C. S. Chu and P. M. Ho, JHEP **0604** (2006) 013 [arXiv:hep-th/0602054].

[22] T. Ishino and N. Ohta, Phys. Lett. B **638** (2006) 105 [arXiv:hep-th/0603215].

[23] C. S. Chu and P. M. Ho, JHEP **0802** (2008) 058 [arXiv:0710.2640 [hep-th]].

[24] K. Narayan, Phys. Rev. D **81** (2010) 066005 [arXiv:0909.4731 [hep-th]].

[25] H. Lü, C. N. Pope and K. W. Xu, Mod. Phys. Lett. A **11** (1996) 1785 [arXiv:hep-th/9604058].

[26] O. I. Bogoyavlensky, Commun. Math. Phys. **51** (1976) 201.

[27] A. G. Reyman, M. A. Semenov-Tian-Shansky, in *Dynamical Systems VII* (Springer, 1993, ed. by V. I. Arnol'd and S. P. Novikov), part II, chapter 2, section 3.2.

[28] R. Argurio, F. Englert and L. Houart, Phys. Lett. B **398**, 61 (1997) [arXiv:hep-th/9701042].

[29] N. Ohta, Phys. Lett. B **403**, 218 (1997) [arXiv:hep-th/9702164].

[30] N. Ohta, Phys. Lett. B **558** (2003) 213 [arXiv:hep-th/0301095]; Phys. Lett. B **559** (2003) 270 [arXiv:hep-th/0302140].